Lecture 19: Inverse Matrices

November 17, 2016 11:39 AM

Chapter 17.3 Motivation

From the row and column space algorithms we can see:

- dim(Row(A))=# of leading 1s in any REF of A
- dim(Col(A))=# of leadings 1s in any REF of A

= rank(A)

We also know:

1) Let A be an mxn matrix.
   $Ax = b$ is consistent for every $b \in \mathbb{R}^m$:
   <=> Col(A)= $\mathbb{R}^m$ (the columns span $\mathbb{R}^m$)
   <=> rank(A)= m (# of rows)
2) Let $b \in \mathbb{R}^m$ such that $Ax = b$ is consistent
   $Ax = b$ has a unique solution:
   <=> "the columns are linearly independent"
   <=> rank(A)=n (# of columns)(ie. no free parameters)

Now, consider matrices A of size nxn (square matrices).
Then,
rank(A)=n (# of rows and # of columns)
<=> "the columns form a basis of $\mathbb{R}^n$"
<=> $Ax = b$ is consistent (because rank(A)=n) and has a unique solution for every $b \in \mathbb{R}^n$ (because rank(A)=n=m)

These matrixes will be called **invertible**.

Chapter 18: Matrix Inverses

Let A be an nxn matrix (a square matrix).

Definition

If B is also an nxn matrix such that $AB = I$ and $BA = I$, then B is called an inverse of A, and A is called **invertible**.

Example

$A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$, $B = ?$ such that $AB = I$ (AB is the identity matrix)

$$\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} & \\ & \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$A \quad B \quad = \quad I$$

Let's guess

$B = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

But we still need to check that $BA = I$

$$\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

If B is an inverse of A, we simply write A^{-1}.

$A^{-1} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$

But how did we find B?

Chapter 18.1 Finding an inverse for **2x2** matrices

Lemma: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then:

1) if $ad - bc \neq 0$, then $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
2) if $ad - bc = 0$, then A is not invertible.

ie. first check if *ad*-*bc*=0, and if so then use the formula.

<u>Lemma</u>: Suppose A is an invertible matrix. Then, any linear system $Ax = b$ is consistent and has a unique solution.

PROOF:

<u>Consistent?</u>

Pick the solution $x = A^{-1}b$. This works:

$Ax = A * (A^{-1}b) = (AA^{-1})b = Ib = b$

identity matrix

<u>Unique?</u>

Let $Ax = b$ and $Ay = b$.

$b = Ax \Rightarrow A^{-1}b = A^{-1}Ax = Ix = x$

$b = Ay \Rightarrow A^{-1}b = A^{-1}Ay = Iy = y$

so $x = A^{-1}b = y$

<u>18.2 Algebraic properties of invertible matrices</u>:

Let $k \neq 0$ be a scalar, p an integer, A and C invertible nxn matrices.
Then, the following are also invertible:

1) A^{-1} $\quad (A^{-1})^{-1} = A$
2) A^p $\quad (A^p)^{-1} = (A^{-1})^p$
3) A^T $\quad (A^T)^{-1} = (A^{-1})^T$
4) kA $\quad (kA)^{-1} = \frac{1}{k}A^{-1}$
5) AC $\quad (AC)^{-1} = C^{-1}A^{-1}$

Trick to remember: Put on socks (A), then shoes (C). Take off shoes (C) first then socks(A)

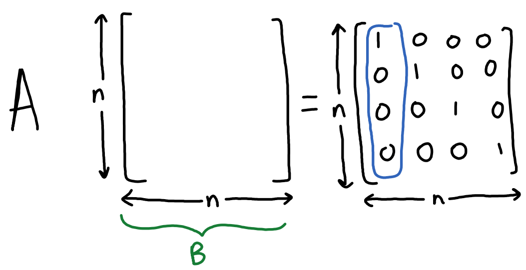
PROOF:

5) $(AC)(C^{-1}A^{-1}) = A(CC^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$
$(C^{-1}A^{-1})(AC) = C^{-1}(A^{-1}A)C = C^{-1}IC = C^{-1}C = I$

<u>18.3 Finding the inverse</u>

Let A be a nxn matrix.

We would like to construct a B such that:



So, $A * first\ column\ of\ B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

The first column of B is the unique solution to $Ax = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 \\ \vdots \end{bmatrix}$

$\lfloor 0 \rfloor$

Similarly, the kth column of B is the unique solution to $Ax = \begin{bmatrix} 0 \\ \vdots \\ k \\ \vdots \\ 0 \end{bmatrix}$ ← kth entry

<u>Example</u>

$A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ $\qquad B = \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} = \begin{bmatrix} 3 & \cdot \\ -2 & \cdot \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$

Solve the system:

$Ax = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\qquad \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \end{array}\right]$

$Ax = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\qquad \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 3 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 1 \end{array}\right]$

Therefore, $B = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$.

<u>This method works, but it's rather long, so we need a shortcut</u>:

Write a super-matrix:

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 3 & -1 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

Keep row reducing until the identity matrix is on the left, and matrix B is on the right.

I ↗ $\qquad$ ↖ matrix B

$AB = I$

<u>Lemma</u>

If $AB = I$, then $BA = I$.